Jacobians and branch points of real analytic open maps

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February 8, 2008

Summary. The main result is that the Jacobian determinant of an analytic open map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ does not change sign. A corollary of the proof is that the set of branch points of f has dimension $\leq n-2$.

Mathematics Subject Classification (2000). Primary 26E05, 26B10; secondary 54C10.

Keywords. Real analytic map, open map, branch points

Abbreviated title: Real analytic open maps

Introduction

The main object of this paper is to prove the following result:

Theorem 1 *The Jacobian of a real analytic open map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ does not change sign.*

One of the referees kindly pointed out that the special case of polynomial maps was proved by Gamboa and Ronga [3]:

Theorem 2 (Gamboa and Ronga) *A polynomial map in $\mathbb{R}^n$ is open if and only if point inverses are finite and the Jacobian does not change sign.*

The proof of Theorem 1 is very similar to methods in [3], which are easily adapted to analytic maps; but as Theorem 1 does not seem to be known, a direct proof may be useful.

$f\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes a (real) analytic map in Euclidean n-space. We always assume f is *open*, that is, f maps open sets onto open sets. Denote the Jacobian matrix of f at $p \in \mathbb{R}^n$ by $df_p = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$. The rank of df_p is called the *rank* of f at p, denoted by $\mathsf{rk}_p f$; the determinant of df_p is the *Jacobian* of f at p, denoted by $Jf(p)$. When the analytic function $Jf\colon \mathbb{R}^n \to \mathbb{R}$ is everywhere non-negative or everywhere non-positive (in a set X), we say Jf *does not change sign (in X)*.

The following sets are defined for any C^1 map $g\colon M \to N$ between n-manifolds (without boundary):

- the set $R_k = \{p \in M\colon \mathsf{rk}_p g \leq k\}$
- the *critical set*, $C = R_{n-1}$
- the *branch set*, $B = \{p \in U\colon g \text{ is not a local homeomorphism at } p\}$

*This research was partially supported by a grant from the National Science Foundation

Note that $B \subset C$ by the inverse function theorem. When g is analytic, we also define:

- the *critical analytic hypersurface* $H \subset C$, comprising those points having a neighborhood in C that is an analytic submanifold of dimension $n-1$
- the *constant rank analytic hypersurface* $V \subset H$, at which $g|H$ has locally constant rank

The following results are byproducts of the proof of Theorem 1:

Theorem 3

(i) *the restricted map* $f|V$ *has rank* $n-1$*,*

(ii) f *is a local homeomorphism at every point of* V*,*

(iii) $B \subset R_{n-2}$*,*

(iv) $\dim R_{n-2} \leq n-2$*.*

When $n=2$, conclusions (iii) and (iv) imply B is a closed discrete set; thus in this case f is *light*, i.e., point inverses are 0-dimensional. From Stoilow [4], which topologically characterizes germs of light open surface maps, we obtain:

Corollary 4 *When* $n=2$*, the germ of* f *at any point is topologically equivalent to the germ at* 0 *of the complex function* z^d *for some integer* $d \neq 0$*.*

A key role in our proofs is played by the following result, Theorem 1.4 of Church [2]:

Theorem 5 (Church) *If* $g\colon \mathbb{R}^n \to \mathbb{R}^n$ *is* C^n *and open with rank* $\geq n-1$ *at every point, then* g *is a local homeomorphism.*

Our results are close to some of those obtained by Church for C^n maps. It is interesting to compare Theorem 3(iii) and Corollary 4 to the following results from paragraphs 1.5 to 1.8 of his paper [2]:

Theorem 6 (Church) *Let* $g\colon M \to N$ *be a* C^n *map between* n*-manifolds.*

(i) *If* $M = N = \mathbb{R}^n$ *and* g *is light, the following conditions are equivalent:*

- **(a)** g *is open*
- **(b)** Jg *does not change sign*
- **(c)** $B \subset R_{n-2}$.

(ii) *If* M *is compact and* g *is open, then* g *is light.*

Proofs

Lemma 7 *Assume the critical set of* f *is* $C = Jf^{-1}(0) = \mathbb{R}^{n-1} \times \{0\}$*, and* $f|C$ *has constant rank* k*,* $0 \leq k \leq n-1$*. Then* f *is a local homeomorphism,* $k = n-1$*, and* Jf *does not change sign in* $\mathbb{R}^n$*.*

Proof It suffices to prove that the conclusion holds in some neighborhood of each point, which we may take to be the origin.

It is convenient to denote points of $\mathbb{R}^n$ as $(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

By the rank theorem we assume that in some open cubical neighborhood N of the origin,

$$f_i(y,0) \equiv 0, \quad i = k+1, \ldots, n \tag{1}$$

Identifying N with $\mathbb{R}^n$ by an analytic diffeomorphism, we assume this holds for all $y \in \mathbb{R}^n$.

Because f is analytic and open, there is a dense open set $\Lambda \subset \mathbb{R}^{n-1}$ such that for every $y \in \Lambda$, the map $t \mapsto f_n(y,t)$ is not constant on any interval. For each $y \in \Lambda$ there exists a maximal integer $\mu(y) \geq 0$ such that

$$0 < j < \mu(y) \implies \left(\frac{\partial}{\partial t}\right)^j f_n\,(y,0) = 0,$$

Fix $y_* \in \Lambda$ such that the function $\mu\colon\ \Lambda \to \mathbb{N}$ takes its minimum value m at y_*. Then $\mu = m$ in a precompact open neighborhood $W \subset \Lambda$ of y_*.

By Taylor's theorem there exists $\epsilon > 0$ such that for (y,t) in the open set

$$N = W\times\ ]-\epsilon,\epsilon\,[\ \subset \mathbb{R}^{n-1}\times\mathbb{R}$$

we have

$$f_n(y,t) = t^m H(y,t), \quad H(y,t) \neq 0 \tag{2}$$

Claim: If $k \leq n-2$ and $(y_0,t_0) \in N$ is such that $f_n(y_0,t_0) = 0$, then $f_{n-1}(y_0,t_0) = 0$. For $t_0 = 0$ by (2), and $k \leq n-2$ implies $f_{n-1}(y_0,0) = 0$ by (1).

Now we assume $k \leq n-2$ and reach a contradiction. Since $f(N)$ is open and contains

$$f(y_*,0) = (a_1,\ldots,a_{n-2},0,0),$$

$f(N)$ also contains points $(a_1\ldots,a_{n-2},\delta,0)$ with $\delta > 0$. But this contradicts the claim.

As f has rank $n-1$ at every point of the critical set, f must be a local homeomorphism by Theorem 5. Therefore for every p, the induced homomoprhism of homology groups

$$\mathbb{Z} = H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{p\}) \to H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{f(p)\}) = \mathbb{Z}$$

is an isomorphism, hence is multiplication by a number $\delta(p) \in \{+1,-1\}$.

Homology theory implies that each of the two level sets of $\delta\colon\ \mathbb{R}^n \to \{+1,-1\}$ is open. As $\mathbb{R}^n$ is connected, $\delta(p)$ is constant. As $\delta(p)$ is the sign of $Jf(p)$ if $Jf(p) \neq 0$, we have proved Jf does not change sign. ∎

Proof of Theorem 1

For any set $Y \subset \mathbb{R}^n$, we say *the local theorem holds in* Y if every point of Y has a neighborhood in Y in which Jf does not change sign.

Lemma 8 *If the local theorem holds in a connected set* Y*, then* Jf *does not change sign in the closure* $\overline{Y}$.

Proof It suffices to prove Jf does not change sign in Y, because Jf is continuous. Define Y_+, Y_- to be the subsets of Y where Jf is respectively ≥ 0 and ≤ 0. These sets are closed in Y by continuity of Jf, and open in Y by hypothesis. As Y is connected, either $Y = Y_+$ or $Y = Y_-$. ∎

The local theorem obviously holds in the set $\mathbb{R}^n \setminus C$ of noncritical points. By Lemma 7, it also holds in relatively open analytic hypersurface $V \subset C$ defined in the introduction. It remains to prove that every point of $C \setminus V$ has a neighborhood in which Jf does not change sign.

Lemma 9 *Every point* $p \in C \setminus V$ *has a neighborhood* $X_p \subset C \setminus V$ *that is an analytic variety of dimension* $\leq n-2$.

Proof Write

$$C \setminus V = (C \setminus H) \cup (H \setminus V)$$

Suppose $p \in C \setminus H$. In this case we take X_p to be the union of the variety C_{sing} of singular points of C and those connected components of $C \setminus C_{\mathsf{sing}}$ having dimension $\leq n-2$.

Suppose $p \in H \setminus V$, or equivalently: $p \in H$ and some minor determinant of *df* vanishes at p but not identically in any neighborhood of p in H. We take X_p to be the intersection of C with the union of the zero sets of such minors. ∎

Now consider any point $p \in C \setminus V$. By Lemma 9, p has a neighborhood $X_p \subset C \setminus V$ that is the union of finitely many smooth submanifolds of $\mathbb{R}^n$ having dimensions $\leq n-2$.

Choose a connected open neighborhood $N_p \subset \mathbb{R}^n$ of p such that $N_p \cap (C \setminus V) = N_p \cap X_p$. Then $N_p \setminus X_p \subset (\mathbb{R}^n \setminus C) \cup V$. Therefore the local theorem holds in $N_p \setminus X_p$.

Now $N_p \setminus X_p$ is connected, by a standard general position argument. Therefore from Lemma 8, with $Y = N_p \setminus X_p$, we infer that Jf does not change sign in $\overline{N_p \setminus X_p}$, which equals $\overline{N_p}$ because X_p is nowhere dense. This completes the proof of Theorem 1.

Proof of Theorem 3

Parts (i) and (ii) of Theorem 2 are proved by applying Lemma 7 locally. Lemma 9 implies (iii), because $B \subset C \setminus V$ by (ii). For (iv), suppose $\dim R_{n-2} = n-1$. Then the variety R_{n-2} contains an analytic hypersurface, which must meet V. As $R_{n-2} \subset C$, this implies $R_{n-2} \cap V \neq \varnothing$, contradicting (i). ∎

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